

Existence and asymptotic behavior for the ground state of quasilinear elliptic equation

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Abstract

In this paper, we are concerned with the existence and asymptotic behavior of minimizers for a minimization problem related to some quasilinear elliptic equations. Firstly, we proved that there exist minimizers when the exponent q equals to the critical case $q^* = 2 + \frac{4}{N}$, which is different from that of [6]. Then, we proved that all minimizers are compact as q tends to the critical case q^* when $a < a^*$ is fixed. Moreover, we studied the concentration behavior of minimizers as the exponent q tends to the critical case q^* for any fixed $a > a^*$.

1 Introduction

In this paper, we consider the following minimization problem

$$d_a(q) = \inf_{u \in M} E_q^a(u) \quad (1.1)$$

where

$$M = \left\{ \int_{\mathbb{R}^N} |u|^2 dx = 1, u \in X \right\},$$

and

$$E_q^a(u) = \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla u|^2 + V(x)|u|^2) dx + \frac{1}{4} \int_{\mathbb{R}^N} |\nabla u^2|^2 dx - \frac{a}{q+2} \int_{\mathbb{R}^N} |u|^{q+2} dx. \quad (1.2)$$

Here, we assume that $0 < q \leq q^* = 2 + \frac{4}{N}$, $a \in \mathbb{R}$ is a constant, the potential $V(x) \in L_{\text{loc}}^\infty(\mathbb{R}^N; \mathbb{R}^+)$. The space X is defined by

$$X = \left\{ u : \int_{\mathbb{R}^N} |\nabla u^2|^2 dx < \infty, u \in H \right\}$$

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with

$$H = \left\{ u : \int_{\mathbb{R}^N} |\nabla u|^2 + V(x)|u|^2 dx < \infty \right\}.$$

Any minimizers of (1.1) solve the following quasilinear elliptic equation

$$-\Delta u - \Delta(u^2)u + V(x)u = \mu u + a|u|^q u, \quad x \in \mathbb{R}^N. \quad (1.3)$$

That is, the Euler-Lagrange equation to problem (1.1), where μ denotes the Lagrange multiplier under the constraint $\|u\|_{L^2}^2 = 1$. Solutions of problem (1.3) also correspond to the standing wave solutions of certain quasilinear Schrödinger equation

$$i\partial_t \varphi = -\Delta \varphi - \Delta(\varphi^2)\varphi + W(x)\varphi - a|\varphi|^q \varphi, \quad x \in \mathbb{R}^N, \quad (1.4)$$

where $\varphi : \mathbb{R} \times \mathbb{R}^N \rightarrow \mathbb{C}$, $W : \mathbb{R}^N \rightarrow \mathbb{R}$ is a given potential. Equation (1.4) arising in several physical phenomena such as the theory of plasma physics, exciton in one-dimensional lattices and dissipative quantum mechanics, see for examples [4, 15, 16, 20] and the references therein for more backgrounds. It is obvious that $e^{-i\mu t}u(x)$ solves (1.4) if and only if $u(x)$ is the solution of equation (1.3).

Equation (1.3) is usually called a semilinear elliptic equation if we ignore the term $-\Delta(u^2)u$. The constrained minimization problem associated to semilinear elliptic equation has been studied widely [2, 9, 10, 11, 12]. The authors considered in [2, 9, 10, 12] the following minimization problem in dimension two:

$$I_a(q) = \inf_{u \in H, \int_{\mathbb{R}^2} |u|^2 dx = 1} J_q^a(u) \quad (1.5)$$

where

$$J_q^a(u) = \frac{1}{2} \int_{\mathbb{R}^2} |\nabla u|^2 dx + \frac{1}{2} \int_{\mathbb{R}^2} V(x)|u|^2 dx - \frac{a}{q+2} \int_{\mathbb{R}^2} |u|^{q+2} dx,$$

and $0 < q \leq q^* = 2$, $a \in \mathbb{R}$ is a constant. By using some rescaling arguments, they obtained that there exists a constant a^* , such that (1.5) has at least one minimizer if and only if $a < a^*$. Moreover, it was discussed in [9, 10, 12] further the concentration and symmetry breaking of minimizers for (1.5) when $q = 2$ and a tends to a^* from left (denoted by $a \nearrow a^*$). Recently, Guo, Zeng and Zhou [11] studied the concentration behavior of minimizers of (1.5) as $q \nearrow 2$ for any fixed $a > a^*$.

There are amount of work considering the existence of solutions for equation (1.3), see [5, 6, 20, 21, 22] for subcritical case and [7, 8, 19, 25, 26] for critical case. By using a constrained minimization argument, for different types of potentials the authors established in [20, 22] the existence of positive solutions of problem (1.3) on the manifold $M = \left\{ \int_{\mathbb{R}^N} |u|^{q+2} = c, u \in X \right\}$ and Nahari manifold when $2 \leq q < \frac{2(N+2)}{N-2}$. In [5, 21], by changing of variables, (1.3) was transformed to a semilinear elliptic equation, then the existence of positive solutions were obtained by mountain pass theorem in Orlicz space or Hilbert space framework. It is worth mentioning that the authors in [6, 13] investigated the following constrained minimization problem associated to the quasilinear elliptic equation (1.3) with $V(x) = \text{constant}$:

$$m(c) = \inf \{ E(u) : |u|_{L^2}^2 = c \}, \quad (1.6)$$

where

$$E(u) = \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u|^2 dx + \frac{1}{4} \int_{\mathbb{R}^N} |\nabla u^2|^2 dx - \frac{1}{q+2} \int_{\mathbb{R}^N} |u|^{q+2} dx. \quad (1.7)$$

They mainly obtained that for any $c > 0$, then

$$m(c) = \begin{cases} -\infty, & \text{if } q > q^* = 2 + \frac{4}{N}, \\ 0, & \text{if } q = q^* = 2 + \frac{4}{N}, \end{cases}$$

and (1.6) possesses no minimizer. On the other hand, when $q \in (0, 2 + \frac{4}{N})$ there holds that $m(c) \in (-\infty, 0]$. Especially, if the energy is strictly less than zero, namely,

$$m(c) \in (-\infty, 0), \quad (1.8)$$

they proved that (1.6) possesses at least one minimizer by using Lions' concentration-compactness principle. In general, condition (1.8) can be verified for any $c > 0$ if $q \in (0, \frac{4}{N})$. But for the case of $q \in (\frac{4}{N}, 2 + \frac{4}{N})$, by setting

$$c(q, N) := \inf\{c > 0 : m(c) < 0\},$$

it was proved in [13] that $c(q, N) > 0$ and (1.6) is achieved if and only if $c \in [c(q, N), +\infty)$. Based on the above results, Jeanjean, Luo and Wang [14] recently discovered that there exists $\hat{c} \in (0, c(q, N))$, such that functional (1.7) admits a local minimum on the manifold $\{u \in X : |u|_{L^2}^2 = c\}$ for all $c \in (\hat{c}, c(q, N))$ and $q \in (\frac{4}{N}, 2 + \frac{4}{N})$. Furthermore, mountain pass type critical point of (1.7) was also obtained therein for all $c \in (\hat{c}, \infty)$, which is different from the minimum solution.

We note that, by taking the scaling $u_c(x) = u(c^{\frac{1}{N}}x)$, then

$$E(u) = c^{1-\frac{2}{N}} \left\{ \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u_c|^2 dx + \frac{1}{4} \int_{\mathbb{R}^N} |\nabla u_c^2|^2 dx - \frac{c^{\frac{2}{N}}}{q+2} \int_{\mathbb{R}^N} |u_c|^{q+2} dx \right\}.$$

It would be easy to see that problem (1.6) can be equivalently transformed to problem (1.1) with $V(x) \equiv \text{constant}$ (Without loss of generality, we assume $V(x) \equiv 0$) by setting $a = c^{\frac{2}{N}}$, namely, the following minimization problem:

$$\tilde{d}_a(q) = \inf_{u \in M} \tilde{E}_q^a(u), \quad (1.9)$$

where $\tilde{E}_q^a(\cdot)$ is given by

$$\tilde{E}_q^a(u) = \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u|^2 dx + \frac{1}{4} \int_{\mathbb{R}^N} |\nabla u^2|^2 dx - \frac{a}{q+2} \int_{\mathbb{R}^N} |u|^{q+2} dx.$$

From the above known results of minimization problem (1.6), we see that (1.9) could be achieved only if $q < q^* = 2 + \frac{4}{N}$. The exponent $q^* = 2 + \frac{4}{N}$ seems to be the critical exponent for the existence of minimizers for (1.9). A natural question one would ask is that does problem (1.1) admit minimizers if $V(x) \not\equiv \text{constant}$? Taking the scaling

$u^\sigma(x) = \sigma^{\frac{N}{2}} u(\sigma x)$, it is easy to know that $E_q^a(u^\sigma) \rightarrow -\infty$ as $\sigma \rightarrow +\infty$ if $q > q^*$ and $V(x) \in L_{\text{loc}}^\infty(\mathbb{R}^N)$. This implies that there is no minimizer for problem (1.1) when $q > q^*$. However, when $q = q^*$, the result is quite different. Indeed, for a class of non-constant potentials, we will prove that there exists a threshold (w.r.t. the parameter a) independent of $V(x)$ for the existence of minimizers for (1.1), see our Theorem 1.1 below for details. Moreover, stimulated by [11], we are further interested in studying the limit behavior of minimizers for (1.1) as $q \nearrow q^*$.

Before stating our main results, we first recall the following sharp Gagliardo-Nirenberg inequality [1]:

$$\int_{\mathbb{R}^N} |u|^{\frac{q+2}{2}} dx \leq \frac{1}{\Upsilon_q} \left(\int_{\mathbb{R}^N} |\nabla u|^2 dx \right)^{\frac{(q+2)\theta_q}{4}} |u|_{L^1}^{\frac{(q+2)(1-\theta_q)}{2}}, \forall u \in \mathcal{D}^{2,1}(\mathbb{R}^N), \quad (1.10)$$

where $1 < \frac{q+2}{2} < \frac{2N}{N-2}$, $\theta_q = \frac{2qN}{(q+2)(N+2)}$, $\Upsilon_q > 0$ and

$$\mathcal{D}^{2,1}(\mathbb{R}^N) \triangleq \{u : \nabla u \in L^2(\mathbb{R}^N), u \in L^1(\mathbb{R}^N)\}.$$

As proved in [1], the optimal constant $\Upsilon_q = \lambda_q a_q$ with

$$\lambda_q = (1 - \theta_q) \left(\frac{\theta_q}{1 - \theta_q} \right)^{\frac{qN}{2(N+2)}} \quad \text{and} \quad a_q = |v_q|_{L^1}^{\frac{q}{N+2}}. \quad (1.11)$$

Here, $v_q \geq 0$ optimizes (1.10) (that is, (1.10) is an identity if $u = v_q$) and is the unique nonnegative radially symmetric solution of the following equation [24]

$$-\Delta v_q + 1 = v_q^{\frac{q}{2}}, \quad x \in \mathbb{R}^N. \quad (1.12)$$

Remark 1.1. *Strictly speaking, it has been proved in [24, Theorem 1.3 (iii)] that v_q has a compact support in \mathbb{R}^N and exactly satisfies a Dirichlet-Neumann free boundary problem. Namely, there exists one $R > 0$ such that v_q is the unique positive solution of*

$$\begin{aligned} -\Delta u + 1 &= u^{\frac{q}{2}}, \\ u &> 0 \text{ in } B_R, u = \frac{\partial u}{\partial n} = 0 \text{ on } \partial B_R. \end{aligned} \quad (1.13)$$

In what follows, if we say that u is a nonnegative solution of a equation like the form of (1.12), we exactly means that u is a solution of the free boundary problem (1.13).

From equation (1.12) and the classical Pohozaev identity, one can prove that

$$\begin{cases} \int_{\mathbb{R}^N} |v_q|^{\frac{q+2}{2}} dx = \frac{1}{1 - \theta_q} \int_{\mathbb{R}^N} |v_q| dx, \\ \int_{\mathbb{R}^N} |\nabla v_q|^2 dx = \frac{\theta_q}{1 - \theta_q} \int_{\mathbb{R}^N} |v_q| dx. \end{cases} \quad (1.14)$$

Using the above notations, we first obtain the following result which addresses the existence of minimizers about problem (1.1) for the critical case of $q = q^*$.

Theorem 1.1. *Let $q = q^* = 2 + \frac{4}{N}$ and a_{q^*} be given by (1.11). Assume that $V(x)$ satisfies*

$$V(x) \in L_{\text{loc}}^\infty(\mathbb{R}^N; \mathbb{R}^+), \quad \inf_{x \in \mathbb{R}^N} V(x) = 0 \quad \text{and} \quad \lim_{|x| \rightarrow \infty} V(x) = \infty. \quad (1.15)$$

Then,

- (i) $d_a(q^*)$ has at least one minimizer if $0 < a \leq a_{q^*}$,
- (ii) There is no minimizer for $d_a(q^*)$ if $a > a_{q^*}$.

Theorem 1.1 is mainly stimulated by [2, Theorem 2.1] and [9, Theorem 1], where the semilinear minimization problem (1.5) was studied. The argument in these two references for studying the non-critical case, namely $a \neq a_{q^*}$ is useful for solving our problem. However, when a equals to the threshold (i.e., $a = a^*$ in their problem), it was proved in [2, 9] that there is no minimizer for problem (1.5). This is quite different from our case since there exists at least one minimizer for (1.1) when $a = a_{q^*}$. This difference is mainly caused by the presence of the extra term $\int_{\mathbb{R}^N} |\nabla u|^2 dx$ in (1.2), which makes the argument in [2, 9] unavailable for studying our problem. To deal with the critical case, we will introduce in Section 2 a suitable auxiliary functional and obtain the boundness of minimizing sequence by contradiction. Then, the existence of minimizers follows directly from the compactness Lemma 2.1.

We remark that if $V(x)$ satisfies condition (1.15), one can easily apply the Gagliardo-Nirenberg inequality (1.10) and Lemma 2.1 to prove that (1.1) possesses minimizers for any fixed $1 < q < q^*$. In what follows, we investigate the limit behavior of minimizers for (1.1) as $q \nearrow q^*$. Firstly, if $a < a_{q^*}$ is fixed, our result shows that the minimizers of (1.1) is compact in the space X as $q \nearrow q^*$. More precisely, we have

Theorem 1.2. *Assume $V(x)$ satisfies (1.15) and let $u_q \in M$ be a nonnegative minimizer of problem (1.1) with $0 < a < a_{q^*}$ and $0 < q < q^* = 2 + \frac{4}{N}$. Then*

$$\lim_{q \nearrow q^*} d_a(q) = d_a(q^*).$$

Moreover, there exists $u_0 \in M$ such that $\lim_{q \nearrow q^} u_q = u_0$ in X , where u_0 is a nonnegative minimizer of $d_a(q^*)$. Here, the sequence $\lim_{q \nearrow q^*} u_q = u_0$ in X means that*

$$u_q \rightarrow u_0 \text{ in } H \text{ and } \int_{\mathbb{R}^N} |\nabla u_q^2|^2 dx \rightarrow \int_{\mathbb{R}^N} |\nabla u_0|^2 dx \text{ as } q \nearrow q^*.$$

On the contrary, if $a > a_{q^*}$, the result is quite different and blow-up will happen in minimizers as $q \nearrow q^*$. Actually, our following theorem tells that all minimizers of (1.1) must concentrate and blow up at one minimal point of the potentials.

Theorem 1.3. *Assume $V(x)$ satisfies (1.15) and $a > a_{q^*}$. Let \bar{u}_q be a non-negative minimizer of (1.1) with $0 < q < q^*$. For any sequence of $\{\bar{u}_q\}$, by passing to subsequence if necessary, then there exists $\{y_{\varepsilon_q}\} \subset \mathbb{R}^N$ and $y_0 \in \mathbb{R}^N$ such that*

$$\lim_{q \nearrow q^*} \varepsilon_q^N \bar{u}_q^2(\varepsilon_q x + \varepsilon_q y_{\varepsilon_q}) = \frac{\lambda^N}{|v_{q^*}|_{L^1}} v_{q^*}(\lambda|x - y_0|) \text{ strongly in } \mathcal{D}^{2,1}(\mathbb{R}^N),$$

where v_{q^*} is the unique nonnegative radially symmetric solution of (1.12) and

$$\lambda = \left(\frac{|v_{q^*}|_{L^1}}{N} \right)^{\frac{1}{N+2}}, \quad \varepsilon_q = \left(\frac{4aq}{q^* \lambda_q a_q (q+2)} \right)^{-\frac{2}{N(q^*-q)}} \xrightarrow{q \nearrow q^*} 0^+. \quad (1.16)$$

Moreover, taking $A := \{x : V(x) = 0\}$, then the sequence $\{y_{\varepsilon_q}\}$ satisfies

$$\text{dist}(\varepsilon_q y_{\varepsilon_q}, A) \rightarrow 0 \quad \text{as } q \nearrow q^*.$$

Throughout the paper, $|u|_{L^p}$ denotes the L^p -norm of function u , C, c_0, c_1 denote some constants.

This paper is organized as follows. In Section 2 we shall prove Theorem 1.1 by some rescaling arguments, especially, we prove that $d_{a^*}(q^*)$ possesses minimizers by introducing an auxiliary minimization problem. Section 3 is devoted to the proof of Theorem 1.2 on the compactness in space X for minimizers of $d_a(q^*)$ as $q \nearrow q^*$. In Section 4, we first establish optimal energy estimates for $d_a(q)$ as $q \nearrow q^*$ for any fixed $a > a_{q^*}$, upon which we then complete the proof of Theorem 1.3 on the concentration behavior of nonnegative minimizers as $q \nearrow q^*$.

2 The existence of minimizers: Proof of Theorem 1.1.

The main purpose of this section is to establish Theorem 1.1. We first introduce the following lemma, which was essentially proved in [23, Theorem XIII.67] and [3, Theorem 2.1], etc.

Lemma 2.1. *Assume $V(x)$ satisfies (1.15), then the embedding from H into $L^p(\mathbb{R}^N)$ is compact for all $2 \leq p < 2^* = \begin{cases} +\infty, & N = 1, 2, \\ \frac{2N}{N-2}, & N \geq 3. \end{cases}$*

Taking $q = q^* = 2 + \frac{4}{N}$ in (1.12), we get $\theta_{q^*} = \frac{N}{N+1}$, $\lambda_{q^*} = \frac{N}{N+1}$ and $a_{q^*} = |v_{q^*}|_{L^1}^{\frac{2}{N}}$. Moreover, (1.14) becomes

$$\begin{cases} \int_{\mathbb{R}^N} |v_{q^*}|^{\frac{q^*+2}{2}} dx = (N+1) \int_{\mathbb{R}^N} |v_{q^*}| dx, \\ \int_{\mathbb{R}^N} |\nabla v_{q^*}|^2 dx = N \int_{\mathbb{R}^N} |v_{q^*}| dx, \end{cases} \quad (2.1)$$

and the Gagliardo-Nirenberg inequality (1.10) can be simply given as

$$\int_{\mathbb{R}^N} |u|^{\frac{q^*+2}{2}} dx \leq \frac{N+1}{Na_{q^*}} \int_{\mathbb{R}^N} |\nabla u|^2 dx \cdot |u|_{L^1}^{\frac{2}{N}}. \quad (2.2)$$

Inspired by the argument of [2, 9], we first prove the following lemma which addresses Theorem 1.1 for the case of $a \neq a_{q^*}$.

Lemma 2.2. *Let $V(x)$ satisfy (1.15) and $q = q^*$. Then*

- (i) $d_a(q^*)$ has at least one minimizer if $0 < a < a_{q^*}$;
- (ii) There is no minimizer for $d_a(q^*)$ if $a > a_{q^*}$.

Proof. (i) If $a < a_{q^*}$, for any $u \in M$, it follows from (2.2) that

$$\int_{\mathbb{R}^N} |u|^{q^*+2} dx \leq \frac{N+1}{Na_{q^*}} \int_{\mathbb{R}^N} |\nabla u^2|^2 dx \left(\int_{\mathbb{R}^N} u^2 dx \right)^{\frac{2}{N}} \leq \frac{N+1}{Na_{q^*}} \int_{\mathbb{R}^N} |\nabla u^2|^2 dx.$$

Thus,

$$\begin{aligned} E_{q^*}^a(u) &= \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla u|^2 + V(x)|u|^2) dx + \frac{1}{4} \int_{\mathbb{R}^N} |\nabla u^2|^2 dx - \frac{a}{q+2} \int_{\mathbb{R}^N} |u|^{q^*+2} dx \\ &\geq \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla u|^2 + V(x)|u|^2) dx + \frac{1}{4} \left(1 - \frac{a}{a_{q^*}} \right) \int_{\mathbb{R}^N} |\nabla u^2|^2 dx. \end{aligned} \quad (2.3)$$

Hence, if $\{u_n\}$ is a minimizing sequence of $d_a(q^*)$ with $a < a_{q^*}$, it is easy to know from above that there exists $C > 0$ independent of n such that

$$\sup_n \int_{\mathbb{R}^N} |\nabla u_n^2|^2 dx \leq C < \infty, \quad \sup_n \|u_n\|_H \leq C < \infty.$$

It then follows from Lemma 2.1 that there exists a subsequence of $\{u_n\}$, denoted still by $\{u_n\}$, and $u \in M$ such that

$$u_n \xrightarrow{n} u \quad \text{in } L^p(\mathbb{R}^N), \quad \forall 2 \leq p < 2^* \quad (2.4)$$

and

$$\int_{\mathbb{R}^N} |\nabla u^2|^2 dx \leq \liminf_{n \rightarrow \infty} \int_{\mathbb{R}^N} |\nabla u_n^2|^2 dx \leq C < \infty.$$

The latter inequality indicates u_n^2 is bounded in $L^{2^*}(\mathbb{R}^N)$. Hence we can deduce from (2.4) that

$$u_n \xrightarrow{n} u \quad \text{in } L^p(\mathbb{R}^N), \quad \forall 2 \leq p < 2 \times 2^*.$$

Therefore,

$$d_a(q^*) = \liminf_{n \rightarrow \infty} E_{q^*}^a(u_n) \geq E_{q^*}^a(u) \geq d_a(q^*).$$

This indicate that

$$u_n \xrightarrow{n} u \quad \text{in } H \quad \text{and} \quad \lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} |\nabla u_n^2|^2 dx = \int_{\mathbb{R}^N} |\nabla u^2|^2 dx.$$

It is means that u is a minimizer of $d_a(q^*)$ for $a < a_{q^*}$.

(ii) Let

$$u_\tau = \frac{\tau^{\frac{N}{2}}}{|v_{q^*}|_{L^1}^{\frac{1}{2}}} \sqrt{v_{q^*}(\tau|x - x_0|)} \quad \text{with some } x_0 \in \mathbb{R}^N.$$

Using (2.1) we have

$$\int_{\mathbb{R}^N} u_\tau^2 dx = \frac{\tau^N}{|v_{q^*}|_{L^1}} \int_{\mathbb{R}^N} |v_{q^*}(\tau x)| dx = \frac{1}{|v_{q^*}|_{L^1}} \int_{\mathbb{R}^N} |v_{q^*}| dx = 1, \quad (2.5)$$

$$\int_{\mathbb{R}^N} |\nabla u_\tau^2|^2 dx = \frac{\tau^{N+2}}{|v_{q^*}|_{L^1}^2} \int_{\mathbb{R}^N} |\nabla v_{q^*}|^2 dx = \frac{N\tau^{N+2}}{|v_{q^*}|_{L^1}}, \quad (2.6)$$

$$\int_{\mathbb{R}^N} u_\tau^{q^*+2} dx = \frac{\tau^{N+2}}{|v_{q^*}|_{L^1}^{2+\frac{2}{N}}} \int_{\mathbb{R}^N} v_{q^*}^{\frac{2(N+1)}{N}} dx = \frac{(N+1)\tau^{N+2}}{|v_{q^*}|_{L^1}^{1+\frac{2}{N}}}, \quad (2.7)$$

$$\int_{\mathbb{R}^N} |\nabla u_\tau|^2 dx = \frac{\tau^2}{|v_{q^*}|_{L^1}} \int_{\mathbb{R}^N} |\nabla \sqrt{v_{q^*}}|^2 dx = c_0 \tau^2, \quad (2.8)$$

and

$$\begin{aligned} \int_{\mathbb{R}^N} V(x) u_\tau^2 dx &= \frac{\tau^N}{|v_{q^*}|_{L^1}} \int_{\mathbb{R}^N} V(x) |v_{q^*}(\tau x)| dx = \frac{1}{|v_{q^*}|_{L^1}} \int_{\mathbb{R}^N} V\left(\frac{x}{\tau} + x_0\right) |v_{q^*}| dx \\ &\rightarrow V(x_0) \quad \text{as } \tau \rightarrow +\infty. \end{aligned} \quad (2.9)$$

If $a > a_{q^*}$, from (2.6) and (2.7) we have

$$\frac{1}{4} \int_{\mathbb{R}^N} |\nabla u_\tau^2|^2 dx - \frac{a}{q^*+2} \int_{\mathbb{R}^N} u_\tau^{q^*+2} dx = \frac{N\tau^{N+2}}{4|v_{q^*}|_{L^1}} \left(1 - \frac{a}{a_{q^*}}\right).$$

This together with (2.8) and (2.9) gives that

$$d_a(q^*) \leq V(x_0) + o(1) + \frac{N\tau^{N+2}}{4|v_{q^*}|_{L^1}} \left(1 - \frac{a}{a_{q^*}}\right) + c_0 \tau^2 \rightarrow -\infty \quad \text{as } \tau \rightarrow +\infty.$$

Therefore, we deduce that $d_a(q^*) = -\infty$ and possesses no minimizer if $a > a_{q^*}$. \square

In view of the above lemma, to complete the proof of Theorem 1.1, it remains to deal with the case of $a = a_{q^*}$. In the following lemma, we first prove there exist minimizers for $d_{a_{q^*}}(q^*)$ when $N \leq 3$.

Lemma 2.3. *Assume that $V(x)$ satisfies (1.15), then $d_a(q^*)$ has at least one minimizer if $a = a_{q^*}$ and $N \leq 3$.*

Proof. Assume $\{u_n\}$ is a minimizing sequence of $d_{a_{q^*}}(q^*)$, similar to the argument of (2.3), it is easy to know that

$$\sup_n \|u_n\|_H \leq C < +\infty.$$

Note that $q^* = 2 + \frac{4}{N} < 2^* - 2$ in view of $N \leq 3$, we thus deduce from the above inequality that

$$\sup_n \int_{\mathbb{R}^N} |u_n|^{q^*+2} dx \leq C < +\infty.$$

This further indicates that

$$\sup_n \int_{\mathbb{R}^N} |\nabla u_n^2|^2 dx \leq C < +\infty.$$

Then similar to the proof (i) of Lemma 2.2, we know that there exists a minimizer of $d_a(q^*)$ and the proof is complete. \square

When $N \geq 4$, the argument of Lemma 2.3 become invalid to obtain the existence of minimizers for $d_{a_{q^*}}(q^*)$ since there holds that $q^* > 2^* - 2$. To deal with this case, we introduce the following auxiliary minimization problem

$$m(c) = \inf \{ F(u); \int_{\mathbb{R}^N} |u| dx = c, u \in \mathcal{D}^{2,1}(\mathbb{R}^N) \}, \quad (2.10)$$

where

$$F(u) = \int_{\mathbb{R}^N} |\nabla u|^2 dx - \frac{Na_{q^*}}{N+1} \int_{\mathbb{R}^N} |u|^{2+\frac{2}{N}} dx.$$

Lemma 2.4. (i) $m(c) = 0$ if $0 < c \leq 1$; $m(c) = -\infty$ if $c > 1$.

(ii) problem (2.10) possesses a minimizer if and only if $c = 1$ and all nonnegative minimizer of $m(1)$ must be of the form

$$\left\{ \frac{\lambda^N v_{q^*}(\lambda|x - x_0|)}{|v_{q^*}|_{L^1}} : \lambda \in \mathbb{R}^+, x_0 \in \mathbb{R}^N \right\}. \quad (2.11)$$

Proof. (i) follows easily by some scaling arguments.

(ii). From (i), we have $m(c) = 0$ for any $c \leq 1$. Thus, if u is a minimizer of $m(c)$ with $c < 1$, we obtain from inequality (2.2) that

$$0 = m(c) \geq \left(1 - c^{\frac{2}{N}}\right) \int_{\mathbb{R}^N} |\nabla u|^2 dx.$$

This implies that $u \equiv 0$, it is a contradiction.

If $c = 1$, one can easily check that any u_0 satisfying (2.11) is a minimizer of $m(1)$. On the other hand, if $u \geq 0$ is a nonnegative minimizer of $m(1)$, we get that

$$\int_{\mathbb{R}^N} |\nabla u|^2 dx = \frac{Na_{q^*}}{N+1} \int_{\mathbb{R}^N} |u|^{2+\frac{2}{N}} dx,$$

which indicates that u is an optimizer of Gagliardo-Nirenberg inequality (2.2). Hence, u must be the form of (2.11). \square

Lemma 2.5. Let $\{u_n\} \subset \mathcal{D}^{2,1}(\mathbb{R}^N)$ be a nonnegative minimizing sequence of $m(1)$, and $\int_{\mathbb{R}^N} |\nabla u_n|^2 dx = 1$. Then, there exists $\{y_n\} \subset \mathbb{R}^N$ and $x_0 \in \mathbb{R}^N$, such that

$$\lim_{n \rightarrow \infty} u_n(x + y_n) = \frac{\lambda_0^N}{|v_{q^*}|_{L^1}} v_{q^*}(\lambda_0|x - x_0|) \quad \text{in } \mathcal{D}^{2,1}(\mathbb{R}^N) \quad \text{with } \lambda_0 = \left(\frac{|v_{q^*}|_{L^1}}{N} \right)^{\frac{1}{N+2}}.$$

Proof. From the definition of $m(1)$ and $\{u_n\}$, one has

$$\int_{\mathbb{R}^N} |u_n|^{2+\frac{2}{N}} dx = \frac{N+1}{Na_{q^*}} + o(1). \quad (2.12)$$

We will prove the compactness of $\{u_n\}$ by Lion's concentration-compactness principle [17, 18].

(I) we first rule out the possibility of *vanishing*: If for any $R > 0$,

$$\limsup_{y \in \mathbb{R}^N} \int_{B_R(y)} |u_n| dx = 0.$$

Then $u_n \xrightarrow{n} 0$ in $L^q(\mathbb{R}^N)$ for any $1 < q < 2^*$, this contradicts (2.12).

(II) Now, if *dichotomy* occurs, i.e., for some $c_1 \in (0, 1)$, there exist $R_0, R_n > 0$, $\{y_n\} \subset \mathbb{R}^N$ and sequences $\{u_{1n}\}, \{u_{2n}\}$ such that

$$\text{supp } u_{1n} \subset B_{R_0}(y_n), \quad \text{supp } u_{2n} \subset B_{R_n}^c(y_n),$$

$$\text{dist}(\text{supp } u_{1n}, \text{supp } u_{2n}) \rightarrow +\infty \text{ as } n \rightarrow \infty,$$

$$\left| \int_{\mathbb{R}^N} |u_{1n}| dx - c_1 \right| \leq \varepsilon, \quad \left| \int_{\mathbb{R}^N} |u_{2n}| dx - (1 - c_1) \right| \leq \varepsilon,$$

$$\liminf_{n \rightarrow \infty} \int_{\mathbb{R}^N} |\nabla u|^2 dx \geq \liminf_{n \rightarrow \infty} \left(\int_{\mathbb{R}^N} |\nabla u_{1n}|^2 dx + \int_{\mathbb{R}^N} |\nabla u_{2n}|^2 dx \right) \geq 0.$$

Let $\tilde{u}_{1n}(x) = u_{1n}(x + y_n)$, then there is $u \in \mathcal{D}^{2,1}(\mathbb{R}^N)$, such that

$$\tilde{u}_{1n}(x) \xrightarrow{n} u \neq 0 \text{ in } L_{loc}^p(\mathbb{R}^N), \quad \forall 1 \leq p < 2^*. \quad (2.13)$$

Since

$$0 = m(1) = F(u_{1n}) + F(u_{2n}) + o_n(1) + \alpha(\varepsilon),$$

where $\alpha(\varepsilon) \rightarrow 0$ as $\varepsilon \rightarrow 0$. Letting $n \rightarrow \infty$ and then $\varepsilon \rightarrow 0$, we then obtain from (2.13) that

$$\int_{\mathbb{R}^N} |u| dx = c_1 < 1 \quad \text{and} \quad 0 \leq F(u) \leq m(1) = 0.$$

This together with (2.2) implies that $u = 0$, it is a contradiction.

(III) The above discussions indicate that *compactness* occurs, i.e., for any $\varepsilon > 0$, $\exists R > 0$ and $\{y_n\} \subset \mathbb{R}^N$, such that if n is large enough,

$$\int_{B_R(y_n)} |u_n| dx \geq 1 - \varepsilon.$$

Then, there exists $u \in \mathcal{D}^{2,1}(\mathbb{R}^N)$ such that

$$u_n(x + y_n) \rightarrow u \text{ in } L^1(\mathbb{R}^N).$$

Therefore, we have $\int_{\mathbb{R}^N} |u| dx = 1$ and

$$m(1) = \lim_{n \rightarrow \infty} F(u_n) \geq F(u) \geq m(1),$$

This indicates that $u \geq 0$ is a minimizer of $m(1)$ and

$$\int_{\mathbb{R}^N} |\nabla u|^2 dx = \lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} |\nabla u_n|^2 dx = 1. \quad (2.14)$$

Moreover, we obtain from Lemma 2.4 (ii) that

$$u(x) = \frac{\lambda^N}{|v_{q^*}|_{L^1}} v_{q^*}(\lambda|x - x_0|),$$

where $\lambda = \left(\frac{|v_{q^*}|_{L^1}}{N} \right)^{\frac{1}{N+2}}$ follows directly from (2.14). This finish the proof of the lemma. \square

Based on the two above lemmas, we are ready to prove that when $a = a_{q^*}$, there exist minimizers for $d_a(q^*)$ for any dimension $N \geq 1$.

Lemma 2.6. *Assume that $V(x)$ satisfies (1.15), then $d_{a_{q^*}}(q^*)$ has at least one minimizer if $a = a_{q^*}$.*

Proof. Let $\{u_n\}$ be a minimizing sequence of $d_{a_{q^*}}(q^*)$. Similar to (2.3), one can deduce from (2.2) that $\{u_n\}$ is bounded in H . Now, we claim that

$$\int_{\mathbb{R}^N} |\nabla u_n|^2 dx \text{ is also bounded uniformly as } n \rightarrow \infty. \quad (2.15)$$

Otherwise, if

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} |\nabla u_n|^2 dx = +\infty.$$

Since $\{u_n\}$ is a minimizing sequence of $d_{a_{q^*}}(q^*)$, it then follows from above that

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} |u_n|^{4+\frac{4}{N}} dx = +\infty \quad \text{and} \quad \lim_{n \rightarrow \infty} \frac{\int_{\mathbb{R}^N} |\nabla u_n|^2 dx}{\int_{\mathbb{R}^N} |u_n|^{4+\frac{4}{N}} dx} = \frac{Na_{q^*}}{N+1}.$$

Setting

$$w_n = \varepsilon_n^N u_n^2(\varepsilon_n x) \text{ with } \varepsilon_n = \left(\int_{\mathbb{R}^N} |\nabla u_n|^2 dx \right)^{-\frac{1}{N+2}}. \quad (2.16)$$

We have

$$\begin{aligned} \int_{\mathbb{R}^N} |w_n| dx &= \int_{\mathbb{R}^N} |u_n|^2 dx = 1, \\ \int_{\mathbb{R}^N} |\nabla w_n|^2 dx &= \varepsilon_n^{N+2} \int_{\mathbb{R}^N} |\nabla u_n|^2 dx = 1, \end{aligned}$$

and

$$\int_{\mathbb{R}^N} |w_n|^{2+\frac{2}{N}} dx = \varepsilon_n^{N+2} \int_{\mathbb{R}^N} |u_n|^{4+\frac{4}{N}} dx \xrightarrow{n} \frac{N+1}{Na_{q^*}}.$$

Thus,

$$E_{q^*}^{a_{q^*}}(u_n) = \frac{1}{4}\varepsilon_n^{-(N+2)}F(w_n) + \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla u_n|^2 + V(x)u_n^2) dx = d_{a_{q^*}}(q^*) + o(1).$$

Consequently,

$$F(w_n) = 4\varepsilon_n^{N+2} \left[d_{a_{q^*}}(q^*) - \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla u_n|^2 + V(x)u_n^2) dx + o(1) \right] \xrightarrow{n} 0 = m(1).$$

This implies that $\{w_n\}$ is a minimizing sequence of $m(1)$ and $\int_{\mathbb{R}^N} |\nabla w_n|^2 dx = 1$. It then follows from Lemma 2.5 that there exists $\{y_n\} \subset \mathbb{R}^N$ such that $w_n(\cdot + y_n) \xrightarrow{n} w_0 = \frac{\lambda_0^N}{|v_{q^*}|_{L^1}} v_{q^*}(\lambda_0 x)$ in $\mathcal{D}^{2,1}(\mathbb{R}^N)$. However, it follows from (2.16) that

$$\liminf_{n \rightarrow \infty} \varepsilon_n^2 \int_{\mathbb{R}^N} |\nabla u_n|^2 dx = \liminf_{n \rightarrow \infty} \int_{\mathbb{R}^N} |\nabla w_n^{\frac{1}{2}}|^2 dx \geq \int_{\mathbb{R}^N} |\nabla w_0^{\frac{1}{2}}|^2 dx \geq C > 0.$$

This indicates that

$$\int_{\mathbb{R}^N} |\nabla u_n|^2 dx \geq C\varepsilon_n^{-2} \xrightarrow{n} +\infty,$$

which contradicts the fact that $\{u_n\}$ is bounded in H . Thus, claim (2.15) is proved. Furthermore, one can similar to the argument of Lemma 2.2 (i) obtain that $u_n \xrightarrow{n} u$ in X with u being a minimizer of $d_{a_{q^*}}(q^*)$. \square

Proof of Theorem 1.1: Lemma 2.2 together with Lemma 2.6 indicates the conclusions of Theorem 1.1. \square

3 Case of $a < a_{q^*}$: Proof of Theorem 1.2.

The aim of this section is to prove that when $a < a_{q^*}$ is fixed, all minimizers of (1.1) are compact in the space X as $q \nearrow q^*$, which gives the proof of Theorem 1.2.

Proof of Theorem 1.2: For any $\eta(x) \in C_0^\infty(\mathbb{R}^N)$ and $|\eta(x)|_{L^2}^2 = 1$. We can find a constant $C > 0$ independent of q , such that

$$d_a(q) \leq E_q^a(\eta) \leq C < \infty. \quad (3.1)$$

Assume that u_q is a nonnegative minimizer of (1.1), we deduce from (1.10) that

$$\begin{aligned} & \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u_q|^2 dx + \frac{1}{4} \int_{\mathbb{R}^N} |\nabla u_q^2|^2 dx + \frac{1}{2} \int_{\mathbb{R}^N} V(x)|u_q|^2 dx \\ &= d_a(q) + \frac{a}{q+2} \int_{\mathbb{R}^N} |u_q|^{q+2} dx \leq C + \frac{a}{q+2} \frac{1}{\lambda_q a_q} |\nabla u_q^2|_{L^2}^{\frac{2q}{q^*}}. \end{aligned} \quad (3.2)$$

This implies that

$$\frac{1}{4} \int_{\mathbb{R}^N} |\nabla u_q^2|^2 dx \leq C + \frac{a}{q+2} \frac{1}{\lambda_q a_q} |\nabla u_q^2|^{\frac{2q}{q^*}}_{L^2}.$$

We claim that

$$\limsup_{q \nearrow q^*} \int_{\mathbb{R}^N} |\nabla u_q^2|^2 dx \leq C < \infty. \quad (3.3)$$

For otherwise, if

$$\int_{\mathbb{R}^N} |\nabla u_q^2|^2 dx \triangleq M_q \rightarrow \infty \text{ as } q \nearrow q^*.$$

On one hand, we know from (3.2) that

$$M_q \leq C + \frac{a}{q+2} \frac{4}{\lambda_q a_q} |\nabla u_q^2|^{\frac{2q}{q^*}}_{L^2} \leq \frac{1}{2} \left(1 - \frac{a}{a_{q^*}} \right) M_q^{\frac{q}{q^*}} + \frac{a}{q+2} \frac{4}{\lambda_q a_q} M_q^{\frac{q}{q^*}},$$

i.e.,

$$M_q \leq \left[\frac{1}{2} \left(1 - \frac{a}{a_{q^*}} \right) + \frac{a}{q+2} \frac{4}{\lambda_q a_q} \right]^{\frac{q^*}{q^*-q}}. \quad (3.4)$$

On the other hand, from the definitions of λ_q and a_q in (1.11) we get that

$$\frac{1}{q+2} \frac{1}{\lambda_q} \rightarrow \frac{1}{4} \text{ and } a_q \rightarrow a_{q^*} \text{ as } q \nearrow q^*. \quad (3.5)$$

Thus,

$$\lim_{q \nearrow q^*} \left[\frac{1}{2} \left(1 - \frac{a}{a_{q^*}} \right) + \frac{a}{q+2} \frac{4}{\lambda_q a_q} \right] = \left[\frac{1}{2} \left(1 - \frac{a}{a_{q^*}} \right) + \frac{a}{a_{q^*}} \right] < 1.$$

This together with (3.4) implies that

$$M_q \leq \left[\frac{1}{2} \left(1 - \frac{a}{a_{q^*}} \right) + \frac{a}{q+2} \frac{4}{\lambda_q a_q} \right]^{\frac{q^*}{q^*-q}} \rightarrow 0 \text{ as } q \nearrow q^*,$$

this is impossible. Hence, (3.3) is obtained and it is easy to further show that

$$\frac{1}{2} \int_{\mathbb{R}^N} (|\nabla u_q|^2 dx + V(x)|u_q|^2) dx \leq C < \infty.$$

Which means that $\{u_q\}$ is bounded in $H^1(\mathbb{R}^N)$. As a consequence, there exists a subsequence (denoted still by $\{u_q\}$), and $0 \leq u_0 \in H$, such that

$$u_q \rightharpoonup u_0 \text{ in } H; \quad u_q \rightarrow u_0 \text{ in } L^p(\mathbb{R}^N), \quad \forall 2 \leq p < 2^*.$$

By applying Lebesgue's dominated convergence theorem, one can obtain from above that

$$\lim_{q \nearrow q^*} \int_{\mathbb{R}^N} |u_q|^{q+2} dx = \int_{\mathbb{R}^N} |u_0|^{q^*+2} dx.$$

Therefore,

$$\begin{aligned}
\lim_{q \nearrow q^*} d_a(q) &= \lim_{q \nearrow q^*} \left[\frac{1}{2} \int_{\mathbb{R}^N} (|\nabla u_q|^2 + V(x)|u_q|^2) dx + \frac{1}{4} \int_{\mathbb{R}^N} |\nabla u_q^2|^2 dx - \frac{a}{q+2} \int_{\mathbb{R}^N} |u_q|^{q+2} dx \right] \\
&\geq \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla u_0|^2 + V(x)|u_0|^2) dx + \frac{1}{4} \int_{\mathbb{R}^N} |\nabla u_0^2|^2 dx - \frac{a}{q^*+2} \int_{\mathbb{R}^N} |u_0|^{q^*+2} dx \\
&= E_{q^*}(u_0) \geq d_a(q^*).
\end{aligned} \tag{3.6}$$

On the other hand, for any $\varepsilon > 0$, there exists $u_\varepsilon \in X$ and $|u_\varepsilon|_{L^2}^2 = 1$, such that

$$E_{q^*}^a(u_\varepsilon) \leq d_a(q^*) + \varepsilon.$$

Then,

$$\begin{aligned}
\lim_{q \nearrow q^*} d_a(q) &\leq \lim_{q \nearrow q^*} E_q(u_\varepsilon) \\
&\leq \lim_{q \nearrow q^*} \left\{ \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla u_\varepsilon|^2 + V(x)|u_\varepsilon|^2) dx + \frac{1}{4} \int_{\mathbb{R}^N} |\nabla u_\varepsilon^2|^2 dx - \frac{a}{q+2} \int_{\mathbb{R}^N} |u_\varepsilon|^{q+2} dx \right\} \\
&= E_{q^*}^a(u_\varepsilon) + \lim_{q \nearrow q^*} \left\{ \frac{a}{q^*+2} \int_{\mathbb{R}^N} |u_\varepsilon|^{q^*+2} dx - \frac{a}{q+2} \int_{\mathbb{R}^N} |u_\varepsilon|^{q+2} dx \right\} \\
&\leq d_a(q^*) + \varepsilon,
\end{aligned}$$

Letting $\varepsilon \rightarrow 0$, this inequality together with (3.6) gives that

$$\lim_{q \rightarrow q^*} d_a(q) = d_a(q^*) = E_{q^*}^a(u_0).$$

Hence, u_0 is a nonnegative minimizer of $d_a(q^*)$ and $u_q \rightarrow u_0$ in X as $q \nearrow q^*$. \square

4 Case of $a > a_{q^*}$: Proof of Theorem 1.3.

This section is devoted to proving Theorem 1.3 on the blow-up of minimizers for (1.1) as $q \nearrow q^*$ for the case of $a > a_{q^*}$. For this purpose, we introduce the following functional

$$\tilde{E}_q^a(u) = \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u|^2 dx + \frac{1}{4} \int_{\mathbb{R}^N} |\nabla u^2|^2 dx - \frac{a}{q+2} \int_{\mathbb{R}^N} |u|^{q+2} dx$$

and consider the following minimization problem

$$\tilde{d}_a(q) = \inf_{u \in M} \tilde{E}_q^a(u). \tag{4.1}$$

We remark that when $a > a_{q^*}$, it follows from (3.5) that

$$\lim_{q \nearrow q^*} \frac{4aq}{q^* \lambda_q a_q (q+2)} = \frac{a}{a_{q^*}} > 1. \tag{4.2}$$

We then obtain from Lemma 4.1 below that $\tilde{d}_a(q) < 0$ if $q < q^*$ and is closed to q^* . As a consequence, one can use [14, Lemma 1.1] to deduce that (4.1) has at least one minimizer.

4.1 The blow-up analysis for the minimizers of (4.1).

In this subsection, we study the following concentration phenomena for the minimizers of (4.1) as $q \nearrow q^*$, which is crucial for the proving of Theorem 1.3.

Theorem 4.1. *Let $a > a_{q^*}$ and u_q be a nonnegative minimizer of (4.1) with $q < q^*$. For any sequence of $\{u_q\}$, there exists a subsequence, denoted still by $\{u_q\}$, and $\{y_{\varepsilon_q}\} \subset \mathbb{R}^N$ such that the scaling*

$$w_q = \varepsilon_q^{\frac{N}{2}} u_q(\varepsilon_q x + \varepsilon_q y_{\varepsilon_q}) \quad (4.3)$$

satisfies

$$\lim_{q \nearrow q^*} \varepsilon_q^N u_q^2(\varepsilon_q x + \varepsilon_q y_{\varepsilon_q}) = w_0^2 := \frac{\lambda^N}{|v_{q^*}|_{L^1}} v_{q^*} \left(\lambda |x - x_0| \right) \text{ in } \mathcal{D}^{2,1}(\mathbb{R}^N), \quad (4.4)$$

where λ, ε_q is given by (1.16) and $x_0 \in \mathbb{R}^N$. Moreover, there exist positive constants C, μ and R independent of q , such that

$$w_q(x) \leq C e^{-\mu|x|} \text{ for any } |x| > R \text{ as } q \nearrow q^*. \quad (4.5)$$

To prove the above theorem, we first give the following energy estimate of $\tilde{d}_a(q)$.

Lemma 4.1. *Let $a > a_{q^*}$ be fixed. Then,*

$$\tilde{d}_a(q) = -\frac{q^* - q}{4q} \left(\frac{4aq}{q^* a_q \lambda_q (q+2)} \right)^{\frac{q^*}{q^* - q}} (1 + o(1)) (\rightarrow -\infty) \text{ as } q \nearrow q^*.$$

Proof. For any $u \in M$, we obtain from (1.10) that

$$\begin{aligned} \tilde{E}_q^a(u) &= \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u|^2 dx + \frac{1}{4} \int_{\mathbb{R}^N} |\nabla u^2|^2 dx - \frac{a}{q+2} \int_{\mathbb{R}^N} |u|^{q+2} dx \\ &\geq \frac{1}{4} \int_{\mathbb{R}^N} |\nabla u^2|^2 dx - \frac{a}{(q+2)\lambda_q a_q} \left(\int_{\mathbb{R}^N} |\nabla u^2|^2 dx \right)^{\frac{q}{q^*}} \end{aligned} \quad (4.6)$$

Setting

$$\int_{\mathbb{R}^N} |\nabla u^2|^2 dx = t \text{ and } g(t) = \frac{1}{4}t - \frac{a}{(q+2)\lambda_q a_q} t^{\frac{q}{q^*}}, \quad t \in (0, +\infty).$$

It is easy to know that $g(t)$ gets its minimum at a unique point

$$t_q = \left(\frac{4aq}{q^* \lambda_q a_q (q+2)} \right)^{\frac{q^*}{q^* - q}}, \quad (4.7)$$

Hence,

$$g(t) \geq g(t_q) = -\frac{q^* - q}{4q} \left(\frac{4aq}{q^* \lambda_q a_q (q+2)} \right)^{\frac{q^*}{q^* - q}}. \quad (4.8)$$

From (4.6) and (4.8), we know that

$$\tilde{d}_a(q) \geq -\frac{q^* - q}{4q} \left(\frac{4aq}{q^* \lambda_q a_q (q+2)} \right)^{\frac{q^*}{q^* - q}}.$$

This gives the lower bound of $\tilde{d}_a(q)$. We next will prove the upper bound.

Let

$$u_\tau = \frac{\tau^{\frac{N}{2}}}{|v_q|_{L^1}^{\frac{1}{2}}} \sqrt{v_q(\tau x)}, \quad \tau > 0,$$

where $v_q(x)$ is the unique nonnegative solution of (1.12). Then, $\int_{\mathbb{R}^N} u_\tau^2 dx = 1$ and it follows from (1.14) that

$$\begin{aligned} \int_{\mathbb{R}^N} |\nabla u_\tau|^2 dx &= \frac{\tau^{N+2}}{|v_q|_{L^1}^2} \int_{\mathbb{R}^N} |\nabla v_q|^2 dx = \frac{\theta_q \tau^{N+2}}{(1 - \theta_q) |v_q|_{L^1}}, \\ \int_{\mathbb{R}^N} u_\tau^{q+2} dx &= \frac{\tau^{\frac{Nq}{2}}}{|v_q|_{L^1}^{\frac{q+2}{2}}} \int_{\mathbb{R}^N} v_q^{\frac{q+2}{2}} dx = \frac{\tau^{\frac{Nq}{2}}}{(1 - \theta_q) |v_q|_{L^1}^{\frac{q}{2}}}, \\ \int_{\mathbb{R}^N} |\nabla u_\tau|^2 dx &= \frac{\tau^2}{|v_q|_{L^1}} \int_{\mathbb{R}^N} |\nabla \sqrt{v_q}|^2 dx \leq c_1 \tau^2. \end{aligned}$$

Taking $\tau = \left(\frac{(1 - \theta_q) t_q |v_q|_{L^1}}{\theta_q} \right)^{\frac{1}{N+2}}$ with t_q given by (4.7), we then have

$$\begin{aligned} \frac{1}{4} \int_{\mathbb{R}^N} |\nabla u_\tau|^2 dx - \frac{a}{q+2} \int_{\mathbb{R}^N} u_\tau^{q+2} dx &= \frac{\theta_q \tau^{N+2}}{4(1 - \theta_q) |v_q|_{L^1}} - \frac{a}{q+2} \frac{\tau^{\frac{Nq}{2}}}{(1 - \theta_q) |v_q|_{L^1}^{\frac{q}{2}}} \\ &= -\frac{q^* - q}{4q} \left(\frac{4aq}{q^* \lambda_q a_q (q+2)} \right)^{\frac{q^*}{q^* - q}}, \end{aligned}$$

and

$$\int_{\mathbb{R}^N} |\nabla u_\tau|^2 dx \leq c_0 \tau^2 \leq c_1 \left(\frac{4aq}{q^* \lambda_q a_q (q+2)} \right)^{\frac{q^*}{q^* - q} \cdot \frac{2}{N+2}}.$$

Therefore,

$$\tilde{E}_q^a(u_\tau) \leq -\frac{q^* - q}{4q} \left(\frac{4aq}{q^* \lambda_q a_q (q+2)} \right)^{\frac{q^*}{q^* - q}} + c_1 \left(\frac{4aq}{q^* \lambda_q a_q (q+2)} \right)^{\frac{q^*}{q^* - q} \cdot \frac{2}{N+2}}. \quad (4.9)$$

From (4.2) we see that $\left(\frac{4aq}{q^* \lambda_q a_q (q+2)} \right)^{\frac{q^*}{q^* - q}} \rightarrow +\infty$ as $q \nearrow q^*$. This together with (4.9) implies that

$$\tilde{d}_a(q) \leq \tilde{E}_q^a(u_\tau) \leq -\frac{q^* - q}{4q} \left(\frac{4aq}{q^* \lambda_q a_q (q+2)} \right)^{\frac{q^*}{q^* - q}} (1 + o(1)).$$

Which gives the upper bound of $\tilde{d}_a(q)$ and the lemma is proved. \square

Lemma 4.2. *Let $a > a_{q^*}$ be fixed and u_q be a nonnegative minimizer of $\tilde{d}_a(q)$. Then*

$$\int_{\mathbb{R}^N} |\nabla u_q^2|^2 dx \approx \frac{4a}{q+2} \int_{\mathbb{R}^N} u_q^{q+2} dx \approx \left(\frac{4aq}{q^* \lambda_q a_q (q+2)} \right)^{\frac{q^*}{q^*-q}} := t_q \quad (4.10)$$

and

$$\frac{\int_{\mathbb{R}^N} |\nabla u_q|^2 dx}{\int_{\mathbb{R}^N} |\nabla u_q^2|^2 dx} \rightarrow 0 \quad \text{as } q \nearrow q^*. \quad (4.11)$$

Here $a \approx b$ means that $\frac{a}{b} \rightarrow 1$ as $q \nearrow q^*$.

Proof. From (4.6), we have

$$\begin{aligned} \tilde{d}_a(q) &= \tilde{E}_q^a(u_q) \geq \frac{1}{4} \int_{\mathbb{R}^N} |\nabla u_q^2|^2 dx - \frac{a}{(q+2)\lambda_q a_q} \left(\int_{\mathbb{R}^N} |\nabla u_q^2|^2 dx \right)^{\frac{q}{q^*}} \\ &:= g(t) = \frac{1}{4}t - \frac{a}{(q+2)\lambda_q a_q} t^{\frac{q}{q^*}} \quad \text{with } t = \int_{\mathbb{R}^N} |\nabla u_q^2|^2 dx. \end{aligned} \quad (4.12)$$

We first prove that

$$\int_{\mathbb{R}^N} |\nabla u_q^2|^2 dx \approx \left(\frac{4aq}{q^* \lambda_q a_q (q+2)} \right)^{\frac{q^*}{q^*-q}} = t_q \quad \text{as } q \nearrow q^*. \quad (4.13)$$

For otherwise, if it is false, then in subsequence sense there holds that

$$\lim_{q \nearrow q^*} \frac{\int_{\mathbb{R}^N} |\nabla u_q^2|^2 dx}{t_q} = \gamma \in [0, 1) \cup (1, +\infty).$$

If $\gamma \in [0, 1)$, then,

$$\begin{aligned} \lim_{q \nearrow q^*} \frac{g(\gamma t_q)}{g(t_q)} &= \lim_{q \nearrow q^*} \frac{\gamma t_q - \frac{4a}{(q+2)\lambda_q a_q} (\gamma t_q)^{\frac{q}{q^*}}}{t_q - \frac{4a}{(q+2)\lambda_q a_q} t_q^{\frac{q}{q^*}}} = \lim_{q \nearrow q^*} \frac{q^* \gamma^{\frac{q}{q^*}} - q\gamma}{q^* - q} \\ &= \gamma(-\ln \gamma + 1) \in [0, 1). \end{aligned} \quad (4.14)$$

Let $\delta := \gamma(-\ln \gamma + 1) \in [0, 1)$, we then obtain from (4.12) and (4.14) that

$$\tilde{d}_a(q) \geq (1 + o(1))g(\gamma t_q) \geq \frac{1+\delta}{2}g(t_q) = -\frac{1+\delta}{2} \cdot \frac{q^* - q}{4q} \left(\frac{4aq}{q^* \lambda_q a_q (q+2)} \right)^{\frac{q^*}{q^*-q}}.$$

This contradicts Lemma 3.1. Similar to the above argument, one can prove also that $\gamma \in (1, +\infty)$ cannot occur. Thus, (4.13) is proved.

We next try to prove (4.11). On the contrary, if it is false, then $\exists \beta > 0$, such that $\int_{\mathbb{R}^N} |\nabla u_q|^2 dx \geq \frac{\beta}{2} \int_{\mathbb{R}^N} |\nabla u_q^2|^2 dx$. Therefore,

$$\begin{aligned} 0 &\geq \tilde{d}_a(q) = \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u_q|^2 dx + \frac{1}{4} \int_{\mathbb{R}^N} |\nabla u_q^2|^2 dx - \frac{a}{q+2} \int_{\mathbb{R}^N} |u_q|^{q+2} dx \\ &\geq \frac{1+\beta}{4} \int_{\mathbb{R}^N} |\nabla u_q^2|^2 dx - \frac{a}{(q+2)\lambda_q a_q} \left(\int_{\mathbb{R}^N} |\nabla u_q^2|^2 dx \right)^{\frac{q}{q^*}} \\ &\triangleq \tilde{g}(s), \end{aligned} \quad (4.15)$$

by setting $\tilde{g}(s) = \frac{1+\beta}{4}s - \frac{a}{(q+2)\lambda_q a_q} s^{\frac{q}{q^*}}$ with $s = \int_{\mathbb{R}^N} |\nabla u_q^2|^2 dx$. One can easily check that $\tilde{g}(s) \geq \tilde{g}(s_q)$ with $s_q = \left(\frac{4aq}{q^* \lambda_q a_q (q+2)(1+\beta)} \right)^{\frac{q^*}{q^*-q}}$ and

$$\tilde{g}(s_q) = -\frac{(1+\beta)(q^*-q)}{4q} \left(\frac{4aq}{q^* \lambda_q a_q (q+2)(1+\beta)} \right)^{\frac{q^*}{q^*-q}} \triangleq A.$$

However

$$\frac{A}{-\frac{(q^*-q)}{4q} \left(\frac{4aq}{q^* \lambda_q a_q (q+2)} \right)^{\frac{q^*}{q^*-q}}} = (1+\beta) \left(\frac{1}{1+\beta} \right)^{\frac{q^*}{q^*-q}} \rightarrow 0 \text{ as } q \nearrow q^*.$$

This together with (4.15) indicates that

$$\frac{\tilde{d}_a(q)}{-\frac{(q^*-q)}{4q} \left(\frac{4aq}{q^* \lambda_q a_q (q+2)} \right)^{\frac{q^*}{q^*-q}}} \rightarrow 0 \text{ as } q \nearrow q^*,$$

which contradicts Lemma 3.1, and (4.11) is proved.

Finally, taking

$$\frac{\tilde{d}_a(q)}{t_q} = \frac{1}{2t_q} \int_{\mathbb{R}^N} |\nabla u_q|^2 dx + \frac{1}{4t_q} \int_{\mathbb{R}^N} |\nabla u_q^2|^2 dx - \frac{a}{(q+2)t_q} \int_{\mathbb{R}^N} |u_q|^{q+2} dx.$$

we then obtain from (4.11) and (4.13) that

$$\frac{4a}{(q+2)t_q} \int_{\mathbb{R}^N} |u_q|^{q+2} dx \rightarrow 1 \text{ as } q \nearrow q^*.$$

This gives (4.10). The proof of this lemma is finished. \square

Applying Lemmas 4.1 and 4.2, we end this subsection by proving Theorem 4.1.

Proof of Theorem 4.1: Set

$$\varepsilon_q = t_q^{-\frac{1}{N+2}} = t_q^{-\frac{2}{Nq^*}} \text{ with } t_q \text{ given by (4.7),}$$

it follows from (4.2) that $\lim_{q \nearrow q^*} \varepsilon_q = 0$. Let u_q be a nonnegative minimizer of $\tilde{d}_a(q)$ and define

$$\tilde{w}_q = \varepsilon_q^{\frac{N}{2}} u_q(\varepsilon_q x).$$

From Lemma 4.2, we have

$$\int_{\mathbb{R}^N} |\nabla \tilde{w}_q^2|^2 dx = \varepsilon_q^{N+2} \int_{\mathbb{R}^N} |\nabla u_q^2|^2 dx \approx t_q^{-1} t_q = 1, \quad (4.16)$$

$$\int_{\mathbb{R}^N} |\tilde{w}_q|^{q+2} dx = \varepsilon_q^{\frac{qN}{2}} \int_{\mathbb{R}^N} |u_q|^{q+2} dx \approx \frac{q^* + 2}{4a_{q^*}}, \quad (4.17)$$

and

$$\int_{\mathbb{R}^N} |\nabla \tilde{w}_q|^2 dx = \varepsilon_q^2 \int_{\mathbb{R}^N} |\nabla u_q|^2 dx = o(1) \varepsilon_q^{-N}. \quad (4.18)$$

Since u_q is a minimizer of $\tilde{d}_a(q)$, then there exists $\mu_q \in \mathbb{R}$, such that

$$-\triangle u_q - \triangle(u_q^2)u_q = \mu_q u_q + a|u_q|^q u_q. \quad (4.19)$$

Therefore,

$$\begin{aligned} \mu_q &= \int_{\mathbb{R}^N} |\nabla u_q|^2 dx + \int_{\mathbb{R}^N} |\nabla u_q^2|^2 dx - a \int_{\mathbb{R}^N} |u_q|^{q+2} dx \\ &= 4\tilde{d}_a(q) - \int_{\mathbb{R}^N} |\nabla u_q|^2 dx + \frac{a(2-q)}{q+2} \int_{\mathbb{R}^N} |u_q|^{q+2} dx. \end{aligned}$$

From Lemmas 4.1 and 4.2, we get that

$$\mu_q \varepsilon_q^{N+2} = \mu_q t_q^{-1} = -\frac{q^* - q}{4q} + o(1) + \frac{2-q}{4}(1 + o(1)) \rightarrow -\frac{1}{N} \text{ as } q \nearrow q^*. \quad (4.20)$$

Using (4.17) and [17, Lemma I.1], we see that there exists $\{y_{\varepsilon_q}\} \subset \mathbb{R}^N$ and $R, \eta > 0$, s.t.

$$\liminf_{q \nearrow q^*} \int_{B_R(y_{\varepsilon_q})} |\tilde{w}_q|^2 dx > \eta > 0.$$

Let

$$w_q = \tilde{w}_q(x + y_{\varepsilon_q}) = \varepsilon_q^{\frac{N}{2}} u_q(\varepsilon_q x + \varepsilon_q y_{\varepsilon_q}), \quad (4.21)$$

then,

$$\liminf_{q \nearrow q^*} \int_{B_R(0)} |w_q|^2 dx > \eta > 0. \quad (4.22)$$

From (4.19), we see that $w_q(x)$ satisfies

$$-\varepsilon_q^N \triangle w_q - \triangle(w_q^2)w_q = \mu_q \varepsilon_q^{N+2} w_q + a \varepsilon_q^{N+2-\frac{Nq}{2}} w_q^{q+1}. \quad (4.23)$$

Note that $N+2 = \frac{Nq^*}{2}$, we can deduce that

$$\varepsilon_q^{N+2-\frac{Nq}{2}} = \varepsilon_q^{\frac{N(q^*-q)}{2}} = t_q^{-\frac{q^*-q}{q^*}} = \left(\frac{4aq}{q^* \lambda_q a_q (q+2)} \right)^{-1} = \frac{q^* \lambda_q a_q (q+2)}{4aq}.$$

Consequently,

$$\lim_{q \nearrow q^*} a \varepsilon_q^{N+2-\frac{Nq}{2}} = \lim_{q \nearrow q^*} \frac{q^* \lambda_q a_q}{q} = a_{q^*}. \quad (4.24)$$

Moreover, for any $\varphi \in C_c^\infty(\mathbb{R}^N)$, we deduce from (4.18) and (4.21) that

$$\left| \varepsilon_q^N \int_{\mathbb{R}^N} \nabla w_q \nabla \varphi dx \right| \leq C \varepsilon_q^N \left(\int_{\mathbb{R}^N} |\nabla w_q|^2 dx \right)^{\frac{1}{2}} = o(1) \varepsilon_q^{\frac{N}{2}} \rightarrow 0 \text{ as } q \nearrow q^*.$$

By passing to subsequence, it then follows from (4.20)-(4.24) that

$$w_q^2 \rightharpoonup w_0^2 \text{ in } \mathcal{D}^{2,1}(\mathbb{R}^N) \text{ as } q \nearrow q^*,$$

where $0 \leq w_0 \not\equiv 0$ satisfies

$$-\Delta(w_0^2)w_0 = -\frac{1}{N}w_0 + a_{q^*}w_0^{3+\frac{4}{N}},$$

i.e.

$$-\Delta(w_0^2) = -\frac{1}{N} + a_{q^*}(w_0^2)^{1+\frac{2}{N}}. \quad (4.25)$$

Using classical Pohozaev identities, we obtain that

$$\int_{\mathbb{R}^N} |\nabla w_0^2|^2 dx = \int_{\mathbb{R}^N} w_0^2 dx \text{ and } \int_{\mathbb{R}^N} (w_0^2)^{\frac{q^*+2}{2}} dx = \frac{N+1}{Na_{q^*}} \int_{\mathbb{R}^N} w_0^2 dx.$$

Recalling the Gagliardo-Nirenberg inequality (2.2), we then have

$$\frac{Na_{q^*}}{N+1} \leq \frac{\int_{\mathbb{R}^N} |\nabla w_0^2|^2 dx (\int_{\mathbb{R}^N} w_0^2 dx)^{\frac{2}{N}}}{\int_{\mathbb{R}^N} (w_0^2)^{2+\frac{2}{N}} dx} = \frac{Na_{q^*}}{N+1} \left(\int_{\mathbb{R}^N} w_0^2 dx \right)^{\frac{2}{N}}. \quad (4.26)$$

This indicates that

$$\int_{\mathbb{R}^N} w_0^2 dx \geq 1.$$

On the other hand, there always holds that

$$\int_{\mathbb{R}^N} w_0^2 dx \leq \liminf_{q \nearrow q^*} \int_{\mathbb{R}^N} w_q^2 dx = 1.$$

Consequently, we have

$$\int_{\mathbb{R}^N} w_0^2 dx = 1, \quad (4.27)$$

and thus

$$w_q \rightarrow w_0 \text{ in } L^2(\mathbb{R}^N) \text{ as } q \nearrow q^*.$$

It then follows from (4.16), (4.23) and (4.25) that

$$\liminf_{q \nearrow q^*} \int_{\mathbb{R}^N} |\nabla w_q^2|^2 dx = \int_{\mathbb{R}^N} |\nabla w_0^2|^2 dx = 1. \quad (4.28)$$

This means that

$$w_q^2 \rightarrow w_0^2 \text{ in } \mathcal{D}^{2,1}(\mathbb{R}^N).$$

Moreover, it follows from (4.26) and (4.27) that $w_0^2 \geq 0$ is an optimizer of (2.2), thus it must be of the form

$$w_0^2 = \frac{\lambda^N}{|v_{q^*}|_{L^1}} v_{q^*}(\lambda|x - x_0|),$$

where $\lambda = \left(\frac{|v_{q^*}|_{L^1}}{N}\right)^{\frac{1}{N+2}}$ follows from (4.28). This completes the proof of (4.4).

Now, it remains to prove (4.5) to complete the proof of Theorem 4.1. Indeed, from (4.23) and (4.24) we see that

$$-\Delta(w_q^2) \leq c(x)w_q^2 \quad \text{with } c(x) = 2a_{q^*}w_q^{q-2}.$$

Similar to the proof of [11, Theorem 1.1], one can use DeGiorgi-Nash-Moser theory as well as the comparison principle to deduce that there exists $C, \beta, R > 0$ independent of q , such that

$$w_q^2(x) \leq Ce^{-\beta|x|} \quad \text{for any } |x| > R \quad \text{as } q \nearrow q^*.$$

This gives (4.5) by taking $\mu = \frac{\beta}{2}$. □

4.2 Proof of Theorem 1.3.

This subsection is devoted to proving Theorem 1.3 on the blow-up behavior of minimizers for (1.1) as $q \nearrow q^*$. We first give precise energy estimates of $d_a(q)$ in the following lemma.

Lemma 4.3. *Let $a > a_{q^*}$ be fixed and $\bar{u}_q(x)$ be a nonnegative minimizer of $d_a(q)$. Then,*

$$0 \leq d_a(q) - \tilde{d}_a(q) \rightarrow 0 \quad \text{as } q \nearrow q^*, \quad (4.29)$$

and

$$\int_{\mathbb{R}^N} V(x)\bar{u}_q^2 dx \rightarrow 0 \quad \text{as } q \nearrow q^*. \quad (4.30)$$

Proof. Let $\varphi(x)$ be a cut-off function such that $\varphi(x) \equiv 1$ if $|x| < 1$ and $\varphi(x) \equiv 0$ if $|x| > 1$. As in Subsection 4.1, we still denote u_q to be a nonnegative minimizer of $\tilde{d}_a(q)$ and let w_q be given by (4.3). For any $x_0 \in \mathbb{R}^N$, we set

$$\tilde{u}_q(x) = A_q \varphi(x - x_0) \varepsilon_q^{-\frac{N}{2}} w_q\left(\frac{x - x_0}{\varepsilon_q}\right) = A_q \varphi(x - x_0) u_q(x - x_0 + \varepsilon_q y_{\varepsilon_q}),$$

where $A_q \geq 1$ such that $\int_{\mathbb{R}^N} \tilde{u}_q^2 dx \equiv 1$. Using the exponential decay of w_q in (4.5), we have

$$0 \leq A_q^2 - 1 = \frac{\int_{|x| \geq 1} \varphi(\varepsilon_q x) w_q^2(x) dx}{\int_{\mathbb{R}^N} \varphi(\varepsilon_q x) w_q^2(x) dx} \leq Ce^{-\frac{\mu}{\varepsilon_q}} \quad \text{as } q \nearrow q^*, \quad (4.31)$$

$$\begin{aligned} \int_{\mathbb{R}^N} V(x) \tilde{u}_q^2(x) dx &= A_q^2 \int_{\mathbb{R}^N} V(\varepsilon_q x + x_0) \varphi(\varepsilon_q x) w_q^2 dx \\ &\rightarrow V(x_0) \int_{\mathbb{R}^N} w_0^2 dx = V(x_0) \quad \text{as } q \nearrow q^*, \end{aligned} \quad (4.32)$$

and

$$\int_{\mathbb{R}^N} |\tilde{u}_q|^{q+2} dx = \varepsilon_q^{-\frac{Nq}{2}} A_q^{q+2} \int_{\mathbb{R}^N} \varphi^{q+2}(\varepsilon_q x) |w_q|^{q+2} dx = \varepsilon_q^{-\frac{Nq}{2}} \int_{\mathbb{R}^N} |w_q|^{q+2} dx + O(e^{-\frac{\mu}{\varepsilon_q}})$$

$$= \int_{\mathbb{R}^N} |u_q|^{q+2} dx + O(e^{-\frac{\mu}{\varepsilon_q}}) \quad \text{as } q \nearrow q^*. \quad (4.33)$$

Similar to the above argument, one can also prove that

$$\int_{\mathbb{R}^N} |\nabla \tilde{u}_q|^2 dx = \int_{\mathbb{R}^N} |\nabla u_q|^2 dx + O(e^{-\frac{\mu}{\varepsilon_q}}) \quad \text{as } q \nearrow q^*, \quad (4.34)$$

and

$$\int_{\mathbb{R}^N} |\nabla \tilde{u}_q|^2 dx = \int_{\mathbb{R}^N} |\nabla u_q|^2 dx + O(e^{-\frac{\mu}{\varepsilon_q}}) \quad \text{as } q \nearrow q^*. \quad (4.35)$$

Therefore, choosing $x_0 \in \mathbb{R}^N$ such that $V(x_0) = 0$, we then deduce from the above estimates that

$$\begin{aligned} 0 &\leq d_a(q) - \tilde{d}_a(q) \leq E_q^a(\tilde{u}_q(x)) - \tilde{E}_q^a(u_q(x)) \\ &= \tilde{E}_q^a(\tilde{u}_q(x)) - \tilde{E}_q^a(u_q(x)) + \frac{1}{2} \int_{\mathbb{R}^N} V(x) \tilde{u}_q^2(x) dx \\ &= \frac{1}{2} V(x_0) + O(e^{-\frac{\mu}{\varepsilon_q}}) + o(1) \rightarrow 0 \quad \text{as } q \nearrow q^*. \end{aligned}$$

Moreover, if \bar{u}_q is a nonnegative minimizer of $d_a(q)$. Then

$$\int_{\mathbb{R}^N} V(x) \bar{u}_q^2 dx = d_a(q) - \tilde{E}_q^a(\bar{u}_q) \leq d_a(q) - \tilde{d}_a(q) \rightarrow 0 \quad \text{as } q \nearrow q^*.$$

□

Proof of Theorem 1.3: Now we still denote \bar{u}_q be a nonnegative minimizer of $d_a(q)$. Applying Lemma 4.3, one can check that all the conclusions in Lemma 4.2 also holds for \bar{u}_q , i.e.,

$$\int_{\mathbb{R}^N} |\nabla \bar{u}_q|^2 dx \approx \frac{4a}{q+2} \int_{\mathbb{R}^N} \bar{u}_q^{q+2} dx \approx \left(\frac{4aq}{q^* \lambda_q a_q (q+2)} \right)^{\frac{q^*}{q^*-q}} = t_q \quad (4.36)$$

and

$$\frac{\int_{\mathbb{R}^N} |\nabla \bar{u}_q|^2 dx}{\int_{\mathbb{R}^N} |\nabla \bar{u}_q|^2 dx} \rightarrow 0 \quad \text{as } q \nearrow q^*. \quad (4.37)$$

Moreover, similar to (4.22), one can prove that there exists $\{y_{\varepsilon_q}\} \subset \mathbb{R}^N$ such that the scaling

$$\bar{w}_q(x) := \varepsilon_q^{\frac{N}{2}} \bar{u}_q(\varepsilon_q x + \varepsilon_q y_{\varepsilon_q})$$

satisfies

$$\liminf_{q \nearrow q^*} \int_{B_R(0)} |\bar{w}_q|^2 dx > \eta > 0.$$

Then, repeating the proof of Theorem 4.1, we can prove that

$$\bar{w}_q^2 \rightarrow w_0^2 := \frac{\lambda^N}{|v_{q^*}|_{L^1}} v_{q^*} \left(\lambda |x - x_0| \right) \quad \text{in } \mathcal{D}^{2,1}(\mathbb{R}^N) \quad \text{with } \lambda = \left(\frac{|v_{q^*}|_{L^1}}{N} \right)^{\frac{1}{N+2}}.$$

Moreover, by (4.30) we see that

$$\int_{\mathbb{R}^N} V(x) \bar{u}_q^2 dx = \int_{\mathbb{R}^N} V(\varepsilon_q x + \varepsilon_q y_{\varepsilon_q}) \bar{w}_q(x) dx \rightarrow 0 \text{ as } q \nearrow q^*.$$

This further indicates that the sequence $\{\varepsilon_q y_{\varepsilon_q}\}$ satisfies

$$\varepsilon_q y_{\varepsilon_q} \rightarrow A = \{x : V(x) = 0\} \text{ as } q \nearrow q^*.$$

The proof of Theorem 1.3 is complete. \square

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